A Mixed Linear and Graded Logic:

Proofs, Terms, and Models

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Overview

Linear Logic

$$A \vdash A$$

$$A,\ B \nvdash A$$

$$A \nvdash A \wedge A$$

Background: Logics

Linear Logic

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Linear Logic + !

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$$\Box_2 A \vdash A \wedge A$$

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Graded Logic

Structural rules are governed by a semiring—the additive part of the semiring is involved in weakening and contraction, and the multiplicative part in usage and composition

A Mixed Linear and Non-Linear Logic

LNL showed that the ! modality could be split into two modalities, Lin and Mny, connecting Linear and Intuitionistic Logic into one system. [1]

[1] Benton CSL 1994 A Mixed Linear and Non-Linear Logic: Proofs, Terms and Models

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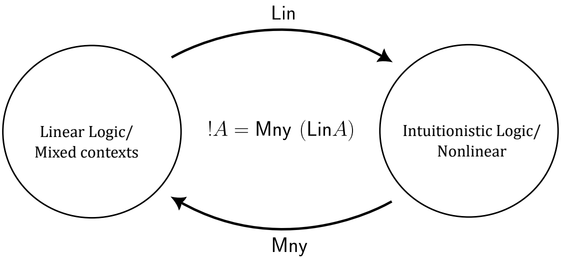

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The Questions

Can this result about the of-course modality, !, and linear logic be carried over to the graded necessity modality, $\Box_r$, and graded logic?

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If so, which (graded) logics would have a similar mutual embedding?

Are those logics even meaningful/useful?

The Sequent Calculus

Our Logic: mGL

Mixed Graded/Linear (mGL) Logic

We present a logic in two fragments connected by modalities that resolve into the graded necessity modality, $\Box_r$.

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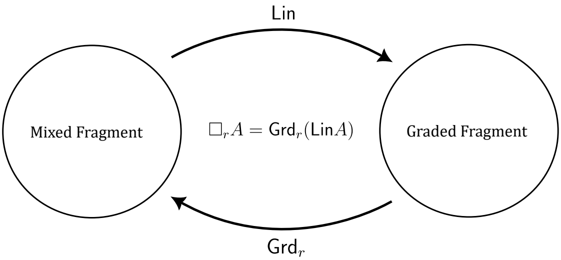

Definition (Context of Formulas)

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And contexts Γ are sequences of linear formulas:

$$\Gamma := \emptyset \mid \Gamma, x : A$$

Definition (Graded Contexts)

Suppose $(\mathcal{R}, 1, *, \mathsf{0}, +, \leq)$ is a preordered semiring.

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We lift the operations of semirings to grade vectors.
A *graded context* $\delta \odot \Delta$ is a pairing of a grade vector and a context

$$\emptyset \odot \emptyset = \emptyset \qquad (\delta, r) \odot (\Delta, x : X) = (\delta \odot \Delta), x : (r \odot X)$$

where $r \odot x$ syntactically pairs a formula with a grade r.

mGL: Graded Fragment

$$(\text{Graded}) X, Y, Z ::= \mathsf{J} \mid X \boxtimes Y \mid \mathsf{Lin}\, A$$

$$\delta \odot \Delta \vdash_{\mathsf{GS}} t : X$$

mGL

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mGL: Mixed Fragment

$$(\text{Linear})A, B, C ::= \mathsf{I} \mid A \otimes B \mid A \multimap B \mid \mathsf{Grd}_r\, X$$

$$\delta \odot \Delta; \Gamma \vdash_{\mathsf{MS}} l : A$$

mGL infrence rules

GST-TenL

$$\frac{(\delta_1, r, r, \delta_2) \odot (\Delta_1, x : X, y : Y, \Delta_2) \vdash_{\mathsf{GS}} t : Z}{(\delta_1, r, \delta_2) \odot (\Delta_1, z : X \boxtimes Y, \Delta_2) \vdash_{\mathsf{GS}} \mathsf{let}\,(x, y) = z\,\mathsf{in}\,t : Z}$$

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MST-GTenL

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$$\mathsf{Lin}A$$

mGL: Lin modality

GST-LinR

$$\frac{\delta \odot \Delta; \emptyset \vdash_{\mathsf{MS}} l : B}{\delta \odot \Delta \vdash_{\mathsf{GS}} \mathsf{Lin}\, l : \mathsf{Lin}\, B}$$

mGL: Lin modality

MST-LinL

$$\frac{\delta \odot \Delta; (x : A, \Gamma) \vdash_{\mathsf{MS}} l : B}{(\delta, 1) \odot (\Delta, z : \mathsf{Lin}\ A); \Gamma \vdash_{\mathsf{MS}} [\mathsf{Unlin}\ z/x]l : B}$$

mGL: Grd modalities

$$\mathsf{Grd}_r\, X$$

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MST-GrdR

$$\frac{\delta \odot \Delta \vdash_{\mathsf{GS}} t : X}{r * \delta \odot \Delta; \emptyset \vdash_{\mathsf{MS}} \mathsf{Grd}\; r\; t : \mathsf{Grd}_r\, X}$$

mGL: Grd modalities

MST-GrdL

$$\frac{(\delta, r) \odot (\Delta, x : X); \Gamma \vdash_{\mathsf{MS}} l : C}{\delta \odot \Delta; (z : \mathsf{Grd}_r\, X, \Gamma) \vdash_{\mathsf{MS}} \mathsf{let}\, \mathsf{Grd}\, r\, x = z\, \mathsf{in}\, l : C}$$

Graded Logic: Graded modality

$$\Box_r A$$

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Ex: Natural Number Semiring

$(\mathbb{N}, 1, 0, *, +)$

Graded Logic: Graded modality

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$\Box_0 A \quad \Box_1 A \quad \Box_{3987} A$

mGL: Recover graded modality

MST-BoxI

$$\frac{\delta \odot \Delta; \emptyset \vdash_{\mathsf{MS}} l : A}{(r * \delta) \odot \Delta; \emptyset \vdash_{\mathsf{MS}} \mathsf{Grd}\, r\, (\mathsf{Lin}\, l) : \Box_r A}$$

mGL: Example

$$\dfrac{\dfrac{\dfrac{}{1 \odot (x : X) \vdash_{\mathsf{GS}} x : X}\ \textsc{Id}_{\mathsf{GS}} \qquad \dfrac{}{1 \odot (y : X) \vdash_{\mathsf{GS}} y : X}\ \textsc{Id}_{\mathsf{GS}}}{(1,1) \odot (x : X, y : X) \vdash_{\mathsf{GS}} (x, y) : X \boxtimes X}\ \boxtimes_R}{\dfrac{2 \odot (x : X) \vdash_{\mathsf{GS}} (x, x) : X \boxtimes X}{6 \odot (x : X); \emptyset \vdash_{\mathsf{MS}} \mathsf{Grd}\ 3\ (x, x) : \mathsf{Grd}_3(X \boxtimes X)}\ \mathsf{Grd}_R}\ \textsc{Cont}_{\mathsf{GS}}$$

mGL: Metatheory

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There is an algorithm taking any mGL *proof to an* mGL *proof without the use of any cut rule.*

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Every formula occurring in a cut-free proof Π *of a GS judgment* $\delta \odot \Delta \vdash_{\mathsf{GS}} X$ *or MS judgement* $\delta \odot \Delta; \Gamma \vdash_{\mathsf{MS}} A$ *consists of subformulas of occuring in the GS judgment* $\delta \odot \Delta \vdash_{\mathsf{GS}} X$ *or MS judgement* $\delta \odot \Delta; \Gamma \vdash_{\mathsf{MS}} A$ *respectively*

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Equational Theory

We give an equational theory on derivations equating proofs of the same sequent which arise from the graded structure, such as when the proof trees differ, as well as equating diffrent steps of the cut reduction algorithm.

Denotational Semantics

Linear Nonlinear Logic

! is an exponential comonad [2]

$$\epsilon_A : !A \to A \qquad \delta_A : !A \to !!A$$

$$m_{A,B} : !A \otimes !B \to !(A \otimes B) \qquad m_i : I \to !I$$

$$d_A : !A \to !A \otimes !A \qquad e_A : !A \to I$$

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Definition

An *LNL model* is a symmetric monoidal adjunction $\mathsf{Mny} \dashv \mathsf{Lin} : \mathcal{M} \to \mathcal{C}$ between a symmetric monoidal closed category $\mathcal{M}$ and a cartesian closed category $\mathcal{C}$ [3]

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Models of $\Box_r$

$\Box_r$ is a graded comonad [4] [5]

$$\epsilon_A : \Box_1 A \to A \qquad \delta_{r,s,A} : \Box_{r*s} A \to \Box_r \Box_s A$$
$$m_{r,A,B} : \Box_r A \otimes \Box_r B \to \Box_r (A \otimes B) \qquad m_{r,I} : I \to \Box_r I$$
$$d_{r,s,A} : \Box_r A \to \Box_s A \otimes \Box_r A \qquad e_A : I \to \Box_0 A$$

[4] Petricek et al. ICALP 2013 Coeffects: Unified Static Analysis of Context-Dependence

[5] Gaboardi et al. ICFP 2016 Combining effects and Coeffects via Grading.

Resolution of a Graded Comonad [6]

Given a symmetric monoidal adjunction $F \dashv G : B \to A$ and a strict action $\circledast : R \times A \to A$ induce and R graded comonad $T : R \to [B, B]$ defined by $T_r = F(r \circledast (G-))$

[6]Fujii et al. FoSSaCS 2016 Towards a Formal Theory of Graded Monads

mGL: Models

Definition

Suppose $\mathcal{C}$ and $\mathcal{M}$ are symmetric monoidal categories[1], $\mathcal{M}$ closed, and R a preordered semiring.

Then a *Mixed Graded/Linear model* is a symmetric monoidal adjunction $\mathsf{Mny} \dashv \mathsf{Lin} : \mathcal{M} \to \mathcal{C}$ along with an exponential action $\odot : \mathcal{R}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$.

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[1] Note that here $\mathcal{C}$ need only be monoidal as the desired structure arises from the action, $\odot$.

Interpretation

Given a mGL model, $\mathsf{Mny} \dashv \mathsf{Lin} : \mathcal{M} \to \mathcal{C}$ and $\odot : \mathcal{R}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$, mutually define two interpretations $[\![-]\!]^{\mathsf{GS}}$ and $[\![-]\!]^{\mathsf{MS}}$ on types and inductivley on derivations: $[\![X]\!]^{\mathsf{GS}} \in \mathcal{C}$ and $[\![A]\!]^{\mathsf{MS}} \in \mathcal{M}$

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With $[\![\mathsf{Lin}\, A]\!]^{\mathsf{GS}} = \mathsf{Lin}[\![A]\!]^{\mathsf{MS}}$
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Bonus:

$\Box_r A = \mathsf{Grd}_r(\mathsf{Lin} A)$

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If Π_1 *and* Π_2 *are* mGL *derivations with* $\Pi_1 \equiv \Pi_2$ *then* $[\![\Pi_1]\!] = [\![\Pi_2]\!]$.

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Theorem (Completeness)

For mGL *derivations* Π_1, Π_2 *if* $[\![\Pi_1]\!] = [\![\Pi_2]\!]$ *in all mixed graded/linear models, then* $\Pi_1 \equiv \Pi_2$.

Natural Deduction

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Lemma (Substitution)

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Theorem (Interderivability)

Any model of the natural deduction system is a model of the sequent calculus and vice versa

Conclusion

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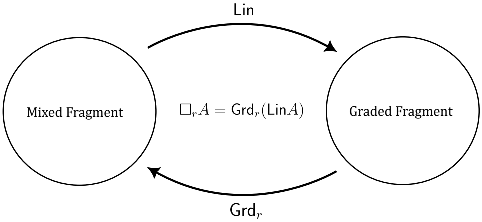

Linear Base

Brunel et al. ESOP '14
Gaboardi et al. ICFP '16
Orchard et al. ICFP '19
Gaboardi et al. POPL '13
Hughes et al. TLLA '21

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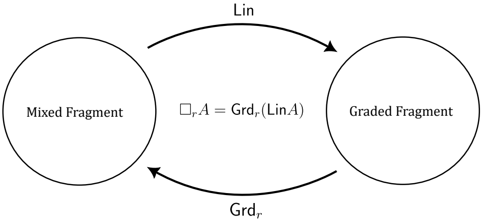

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Graded Base

Petricek et al. ICALP '13
Petricek et al. ICFP '14
Bernardy et al. ICFP '20
Atkey LICS '18
Choudhury et al. POPL '21

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Graded Adjoint Logic

First steps towards a graded adjoint logic

TL;DR

In this work we give a logical rendering of Fujii et al's adjoint decomposition of a graded comonad from the 2016 FoSSaCs paper, Towards a Formal Thoery of Graded Monads.

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In this work we give a logical rendering of Fujii et al's adjoint decomposition of a graded comonad from the 2016 FoSSaCs paper, Towards a Formal Thoery of Graded Monads.

This follows Benton's work on Linear/Nonlinear Logic (LNL) from the CSL 1994 extended abstract, A Mixed Linear and Non-Linear Logic, from which our work derives its name.